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THE GRAHAM CONJECTURE IMPLIES THE ERDŐS-TURÁN CONJECTURE

LIANGPAN LI

Abstract. Erdős and Turán conjectured that any set \( A \subset \mathbb{N} \) with
\[
\sum_{a \in A} 1/a = \infty
\]
should contain infinitely many progressions of arbitrary length \( k \geq 3 \). For the two-dimensional case Graham conjectured that if \( B \subset \mathbb{N} \times \mathbb{N} \) satisfies
\[
\sum_{(a,b) \in B} 1/(a^2 + b^2) = \infty,
\]
then for any \( x \geq 2 \), \( B \) contains an \( x \times x \)-non-parallel grid. In this paper it is shown that if the Graham conjecture is true for some \( x \geq 2 \), then the Erdős-Turán conjecture is true for \( k = 2x - 1 \).

1. Introduction

One famous conjecture of Erdős and Turán [2] asserts that any set \( A \subset \mathbb{N} \) with
\[
\sum_{a \in A} 1/a = \infty
\]
should contain infinitely many progressions of arbitrary length \( k \geq 3 \). There are two important progress towards this direction due to Szemerédi [7] and Green and Tao [5] respectively, which assert that if \( A \) has positive upper density or \( A \) is the set of the prime numbers, then \( A \) contains infinitely many progressions of arbitrary length.

If one considers the similar question in the two-dimensional plane, Graham [3] conjectured that if \( B \subset \mathbb{N} \times \mathbb{N} \) satisfies
\[
\sum_{(a,b) \in B} 1/(a^2 + b^2) = \infty,
\]
then \( B \) contains the four vertices of an axes-parallel square. More generally, for any \( x \geq 2 \), it should be true that \( B \) contains an \( x \times x \)-non-parallel grid. Furstenberg and Katznelson [3] proved the two-dimensional Szemerédi theorem, that is, any set \( B \subset \mathbb{N} \times \mathbb{N} \) with positive upper density contains an \( x \times x \)-non-parallel grid. In another words, such a set \( B \) contains any finite pattern. The purpose of this paper is to show that if the Graham conjecture is true, then the Erdős-Turán conjecture is also true.

2. The Graham Conjecture Implies the Erdős-Turán Conjecture

Suppose that the Erdős-Turán conjecture is false for \( k = 3 \). Then there exists a set
\[
A = \{a_1 < a_2 < a_3 < \cdots \} \subset \mathbb{N}
\]
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\[ \sum_{i=1}^{N} \frac{1}{a_i} = \infty \text{ such that } A \text{ contains no arithmetic progression of length } 3. \]

Define a set \( B \subset \mathbb{N} \times \mathbb{N} \) by

\[ B = \left\{ (n, m, m): n \in \mathbb{N}, m \in \mathbb{N} \right\}. \]

Then

\[ \sum_{(a,b,c) \in B} \frac{1}{a^2 + b^2 + c^2} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(n+m)^2 + n^2} \]

\[ \geq \sum_{n=1}^{\infty} \frac{1}{n^2 + n^2} \]

\[ \geq \sum_{n=1}^{\infty} \frac{1}{n^2} \]

\[ = \sum_{n=1}^{\infty} \frac{1}{n^2} = \infty. \]

In the sequel we indicate that \( B \) contains no square and argue it by contradiction. This would mean that the Graham conjecture is false for \( n = 2 \). Suppose that for some \( n, m, l \in \mathbb{N} \), \( B \) contains a square of the following form:

\[ (a_n, n, m + l), (a_n, n + m, m + l), \]

\[ (a_n, n + m, m), \]

\[ (a_n, n, m), \]

\[ (a_n, n + m, m). \]

It follows easily from the construction of \( B \) that \( a_n - 1 = a_n + 1 \in A \), which yields a contradiction since \( A \) contains no arithmetic progression of length 3 according to the initial assumption.

Similarly, if the Graham conjecture is true for some \( n > 2 \), then the Erdős–Rumsey conjecture is true for \( k = 2 - 1 \). The interested reader can easily provide a proof.

3. Concluding Remarks

Let \( r(n, N) \) be the maximal cardinality of a subset \( A \) of \( \{1, 2, \ldots, N\} \) which is free of \( k \)-term arithmetic progressions. H. Freeman [1] and Rankin [6] had shown that \( r(n, N) \) is \( \Omega(\log N) \) if \( n > 1 \).
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